Probability & Statistics (1)

Axioms of Probability (II)

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2. Probability as a Continuous Set Function
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Sample Spaces Having Equally Likely Outcomes

Axiom 1	$0 \leq P(E) \leq 1$
Axiom 2	$P(S) = 1$
Axiom 3	$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

- **範例一**

今天有個摸彩箱，裡面有編號1到100的彩球，假設抽到每一顆球的機率相等。

(1) 抽一顆球的樣本空間為何?

$$S = \{1,2,3, \dots, 99,100\}$$

(1) 抽到任一顆球的機率為何?

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{100\})$$

$$P(\{i\}) = \frac{1}{N}, i = 1,2,3, \dots, 100$$

Sample Spaces Having Equally Likely Outcomes

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- 對於每個事件E來說，他們出現的機率為:

$$P(E) = \frac{number\ of\ outcomes\ in\ E}{number\ of\ outcomes\ in\ S}$$

Sample Spaces Having Equally Likely Outcomes

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- **範例二**

如果今天擲兩顆骰子，合起來點數要等於8點，請問機率為何?

Solution:

我們可以將所以可能性列出: (2,6), (3,5), (4,4), (5,3), (6,2)等五種。

擲兩顆骰子全部點數的組合數: $\binom{6}{1}\binom{6}{1} = 36$

所以合起來點數為8點的機率為: $\frac{5}{36}$

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- 範例三

今天摸彩箱裡有2個大獎、5個中獎、10個小獎，假設抽到每顆球的機率相等。如果我們要連抽5次，且抽完不放回的前提下，抽中1個大獎+2個中獎+2個小獎的機率為何?

Solution:

$$\frac{\binom{2}{1}\binom{5}{2}\binom{10}{2}}{\binom{17}{5}} = \frac{2 \times 10 \times 45}{17 \times 14 \times 13 \times 2} = \frac{225}{1547}$$

Sample Spaces Having Equally Likely Outcomes

Axiom 1	$0 \le P(E) \le 1$
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- **範例四**

假設今天有10對夫妻(共20人)，現在隨機找5個沒有姻親關係的人。

Solution

看到這個問題，應該會有一個疑問，我們是一次找五個人還是有順序地找五個人? 這兩者機率會是一樣的嗎?

Non-ordered selection	Ordered selection
$P(N) = \dfrac{\binom{10}{5} 2^5}{\binom{20}{5}}$	$P(N) = \dfrac{20 \times 18 \times 16 \times 14 \times 12}{20 \times 19 \times 18 \times 17 \times 16}$

Sample Spaces Having Equally Likely Outcomes

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- **範例五**

假設你今天在役男抽籤現場，應到n位役男前來抽籤，其中有r個籤為海軍陸戰隊，其餘籤為陸軍、空軍與海軍艦艇兵，其中有k位役男需要請公所代抽$(r > k)$。假設代抽一定會抽到海軍陸戰隊且優先一次全部抽完，其餘所有人抽到每個兵種的機率皆相同，請問抽到海軍陸戰隊籤的發生機率為何?

Solution:

$$P(Marine\ Corps) = \frac{\binom{k}{k}\binom{n-k}{r-k}}{\binom{n}{r}}$$

Sample Spaces Having Equally Likely Outcomes

Axiom 1	$0 \leq P(E) \leq 1$
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- **範例六**

今天來玩撲克牌遊戲，抽五張牌。假設每張牌被抽到的機率都相等的前提下，回答下列情境的發生機率:

(1) 抽到鐵支(四條)的機率?

(2) 抽到葫蘆的機率?

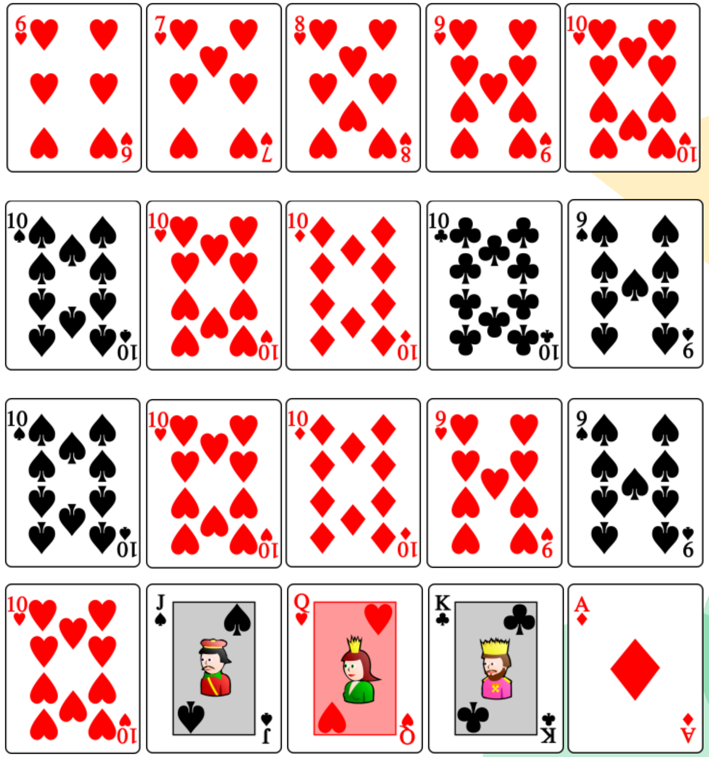
(3) 抽到順子(不含同花)的機率?

(4) 抽到同花順的機率?

如果今天是四個人一起完，你抽中:

(1) 出現一條龍(13種花色都有一張)的機率?

(2) 每個人都至少有一張2個機率為何?



Sample Spaces Having Equally Likely Outcomes

Axiom 1	$0 \le P(E) \le 1$
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- **範例七**

如果今天派對上有n個人，任兩人生日不在同一天的機率有多少?假設不考慮閏年的前提下，一年只有365天，且生日出現在每一天的機率皆相同。請問當多少人以上；此時，任兩人有機會生日在同一天的機率將會小於0.5?

Solution

$$P(D_i \neq D_j, i \neq j) = \frac{365 \times 364 \times 363 \times \cdots \times (365 - n + 1)}{(365)^n}$$

When $n \geq 23$, $P(D_i \neq D_j, i \neq j) < 0.5$

Sample Spaces Having Equally Likely Outcomes

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- **範例八**

某電視台舉辦全明星運動賽的合宿活動(紅藍隊各10人)，每兩兩一組作為室友，請問任一組室友不得同時有紅隊與藍隊的機率為何?

Solution

總共有$\binom{20}{2,2,2,\ldots,2} = \frac{20!}{(2!)^{10}}$的室友配對組合(有考慮排序)。

不過這一題並無需要考慮排序問題，因此共有$\frac{20!}{(2!)^{10}(10!)}$組合。

考量到兩兩室友中，任一方都來自相同的隊伍

$$P(\text{不同隊}) = \frac{\left(\frac{10!}{2^5 5!}\right)^2}{\frac{20!}{2^{10}10!}} = \frac{(10!)^3}{(5!)^2 20!}$$

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- **範例九**

假設班上組共有50人，其中有20參加系籃、15人參加系排、20人參加系羽；同時參加系籃與系排的有7人、同時參加系排與系羽的有10人、同時參加系籃與系羽的有5人、三種同時參加的有3人；試問有多少人至少參加一個以上的系隊?

Solution:

Sample Spaces Having Equally Likely Outcomes

Axiom 1	$0 \leq P(E) \leq 1$
Axiom 2	$P(S) = 1$
Axiom 3	$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

- **範例十**

今天我們辦聖誕節交換禮物活動，總共有N個人參與提供N個獎品抽獎，每個人抽中每個禮物的機率相同，請問沒有任何人抽中自己禮物的機率為何?

Solution:

因為要計算沒有任何人抽中自己禮物的機率不好算，但是計算它的補集就會比較簡單，也就是至少有一個人抽中自己的禮物的機率。

此時，我們就可以定義第i個人抽到自己禮物的事件為$E_i, i = 1,2,\ldots,N$。

Sample Spaces Having Equally Likely Outcomes

Axiom 1	$0 \le P(E) \le 1$
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- **範例十**

所以至少一個人抽到自己禮物的機率就可以利用聯集計算出來:

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{n+1} \sum_{i_1<i_2<\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) + \cdots + (-1)^{N+1} P(E_1E_2\cdots E_N)$$

其中，$P(E_{i_1}E_{i_2}\cdots E_{i_n})$的意思就是有n個人抽到自己禮物的交集，所以就可以被表示為:

$$P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{(N-n)!}{N!}$$

至於前方的$P(E_{i_1}E_{i_2}\cdots E_{i_n})$會有多少組合，那就是$\binom{N}{n}$個。

$$\therefore \sum_{i_1<i_2<\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \binom{N}{n}\frac{(N-n)!}{N!} = \frac{N!}{(N-n)!\,n!}\frac{(N-n)!}{N!} = \frac{1}{n!}$$

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- 範例十

$$\sum_{i_1<i_2<\cdots<i_n} P\left(E_{i_1}E_{i_2}\cdots E_{i_n}\right) = \binom{N}{n}\frac{(N-n)!}{N!} = \frac{N!}{(N-n)!\,n!}\frac{(N-n)!}{N!} = \frac{1}{n!}$$

我們可以得出至少一個人抽到自己的禮物機率為:

$$P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{N+1}\frac{1}{N!}$$

而本題的答案則是「至少一個人抽到自己的禮物機率」的補集:

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!}$$

Probability as a Continuous Set Function

如果我們有一個事件序列(a sequence of events)，假如他屬於遞增序列(an increasing sequence)

$$E_1 \subset E_2 \subset \cdots \subset E_n E_{n+1} \subset \cdots$$

那如果是遞減序列(a decreasing sequence)

$$E_1 \supset E_2 \supset \cdots \supset E_n E_{n+1} \supset \cdots$$

所以$\{E_n, n \geq 1\}$屬於遞增序列，則可以被定義為

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$$

相對地，如果$\{E_n, n \geq 1\}$屬於遞減序列，則可以被定義為:

$$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$$

Probability as a Continuous Set Function

- **Propositions 1**

If $\{E_n, n \geq 1\}$ is either an increasing or a decreasing sequence of events, then

$$\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$$

Proof:

我們假設$\{E_n, n \geq 1\}$為一個遞增序列，另外令某事件$F_n, n \geq 1$

$F_1 = E_1$

$$F_n = E_n E_{n-1}^c = E_n \left(\bigcup_{1}^{n-1} E_i\right)^c = E_n E_{n-1}^c, n > 1$$

Probability as a Continuous Set Function

F_n包含E_n，但卻不包含前面其他的事件$(E_i, i < n)$。

證明F_n事件彼此互斥:

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i \ \ and \ \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \,, \forall n \geq 1$$

$$\therefore P\left(\bigcup_{1}^{\infty} E_i\right) = P\left(\bigcup_{1}^{\infty} F_i\right) = \sum_{1}^{\infty} P(F_i) \, [by \ Axiom \ 3] = \lim_{n\to\infty} \sum_{1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} \sum_{1}^{n} P\left(\bigcup_{1}^{n} F_i\right) = \lim_{n\to\infty} \sum_{1}^{n} P\left(\bigcup_{1}^{n} E_i\right) = \lim_{n\to\infty} P(E_n)$$

Probability as a Continuous Set Function

如果$\{E_n, n \geq 1\}$屬於遞減序列，換句話說，$\{E_n^c, n \geq 1\}$就是遞增序列；因此…

$$P\left(\bigcup_{1}^{\infty} E_i^c\right) = \lim_{n\to\infty} P(E_n^c)$$

$$\because \bigcup_{1}^{\infty} E_i^c = \left(\bigcap_{1}^{\infty} E_i\right)^c$$

$$\therefore P\left(\left(\bigcap_{1}^{\infty} E_i\right)^c\right) = \lim_{n\to\infty} P(E_n^c)$$

$$1 - P\left(\bigcap_{1}^{\infty} E_i\right) = \lim_{n\to\infty}\left[1 - P(E_n)\right] = 1 - \lim_{n\to\infty} P(E_n)$$

$$\therefore P\left(\bigcap_{1}^{\infty} E_i\right) = \lim_{n\to\infty} P(E_n)$$

Probability as a Measure of Belief

- **範例十一**

疫情逐漸趨緩，今天你來到了日本賽馬場賭馬，你隨手拿了一份馬報並且仔細精算一下，勝率如下:

Horse Number and probability of winning							
#1	20%	**#3**	15%	**#5**	10%	**#7**	10%
#2	20%	**#4**	15%	**#6**	10%		

對你來說，賭贏家會是前三批馬其中一批，還是賭贏家會是1,5,6,7其中一匹，哪個比較好?

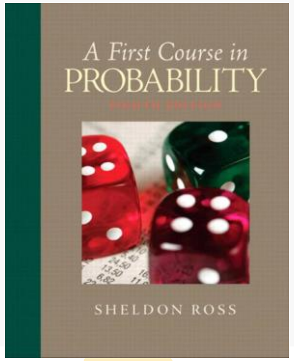
Solution

第一種情況的賭贏機率: $0.2 + 0.2 + 0.15 = 0.55$

第二種情況的賭贏機率: $0.2 + 0.1 + 0.1 + 0.1 = 0.5$

故選第一種情況會比較好!

[#5] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

2. A customer visiting the suit department of a certain store will purchase a suit with probability .22, a shirt with probability .30, and a tie with probability .28. The customer will purchase both a suit and a shirt with probability .11, both a suit and a tie with probability .14, and both a shirt and a tie with probability .10. A customer will purchase all 3 items with probability .06. What is the probability that a customer purchases
(a) none of these items?
(b) exactly 1 of these items?

8. Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that
(a) either A or B occurs?
(b) A occurs but B does not?
(c) both A and B occur?

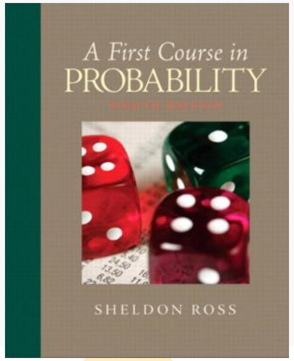
9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

14. The following data were given in a study of a group of 1000 subscribers to a certain magazine: In reference to job, marital status, and education, there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect.
Hint: Let M, W, and G denote, respectively, the set of professionals, married persons, and college graduates. Assume that one of the 1000 persons is chosen at random, and use Proposition 4.4 to show that if the given numbers are correct, then $P(M \cup W \cup G) > 1$.

31. A 3-person basketball team consists of a guard, a forward, and a center.
(a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?
(b) What is the probability that all 3 players selected play the same position?

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

[#5] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

Prove the following relations:

1. $EF \subset E \subset E \cup F$.

6. Let E, F, and G be three events. Find expressions for the events so that, of E, F, and G,

(a) only E occurs;
(b) both E and G, but not F, occur;
(c) at least one of the events occurs;
(d) at least two of the events occur;
(e) all three events occur;
(f) none of the events occurs;
(g) at most one of the events occurs;
(h) at most two of the events occur;
(i) exactly two of the events occur;
(j) at most three of the events occur.

7. Find the simplest expression for the following events:

(a) $(E \cup F)(E \cup F^c)$;
(b) $(E \cup F)(E^c \cup F)(E \cup F^c)$;
(c) $(E \cup F)(F \cup G)$.

12. Show that the probability that exactly one of the events E or F occurs equals $P(E) + P(F) - 2P(EF)$.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

The End

If you have any questions, please do not hesitate to ask me.

Thank you for your attention))